

Black hole entropy and quantum gravity

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Abstract An elementary introduction is given to the problem of black hole entropy as formulated by Bekenstein and Hawking. The information theoretic basis of Bekenstein's formulation is briefly surveyed, and compared with Hawking's approach. The issue of calculating the entropy by actual counting of microstates is taken up next, within two currently popular approaches to quantum gravity, viz., superstring theory and canonical quantum gravity. The treatment of the former essay is confined to a few remarks, mainly of a critical nature, while some computational techniques of the latter approach are elaborated. We conclude by trying to find commonalities between these two rather disparate directions of work.

Keywords Black hole entropy, quantum gravity, computational methods

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1. Introduction

The intriguing possibility that the gravitational force due to a star may be so strong that not even light could escape from it, first appeared in Laplace's analysis [1], almost two hundred years ago. Using the Newtonian formula for the escape velocity of a point mass from a gravitating sphere of mass M and radius R , and setting it equal to c , the velocity of light, Laplace obtained the size of the gravitating sphere to be

$$R = \frac{2GM}{c^2}, \quad (1)$$

a formula that is now well-known to yield the Schwarzschild radius of a black hole of mass M . For $R = R_{\text{sun}}$, the radius of the sun assumed to be a homogeneous sphere with density ρ , one gets $\rho \sim 10^{18}$ gms/cm³. Matter at such a density can hardly be stable under its self-gravity. In fact, we now know that, after exhausting their nuclear fuel, stars that are still heavier than a certain limiting mass (the Chandrasekhar mass) very likely undergo gravitational collapse: all matter (and radiation) inside collapses to a point, forming a spacetime singularity – a black hole from which nothing escapes.

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While gravitational collapse *per se* still defies a complete understanding, black holes have a rather precise description within classical general relativity. They constitute a three parameter family of exact solutions of Einstein's celebrated equation, the three parameters being the mass M , the electric charge Q and the angular momentum $|L|$. The solutions describe spacetime geometries with a unique point at which the (Riemann) curvature becomes singular. In the generic case, however, the singularity is never 'naked' ; it is always enshrouded by a null surface known as the event horizon.

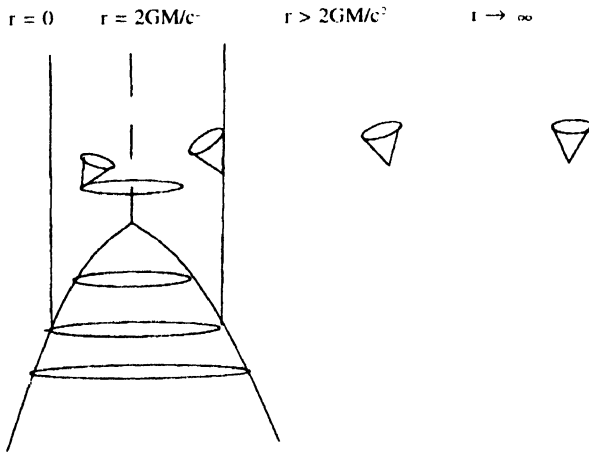


Figure 1. Gravitational collapse of a spherical star

Figure 1 depicts the gravitational collapse of a spherical star. One spatial dimension has been suppressed, so that the ellipses actually represent 2-spheres at different time-slices. The envelope of the ellipses is the spherically collapsing body. When the size of the star shrinks to that of a sphere of radius $R = 2GM/c^2 \equiv R_{Sch}$, the body is barely visible to the external observer. This is delineated in the figure by the tipping of the light cone as one approaches the horizon. At the horizon, the generators of the null cone align with that surface, so that light from the collapsing body grazes it. The local spacetime metric changes signature at the horizon. As the star shrinks further, light from its surface no longer reaches the outside world. All null and time-like geodesics, associated with trajectories of massless and massive particles in the black hole geometry, are inexorably focussed onto the curvature singularity at $r = 0$. An observer on the collapsing body, however, notices nothing special as she crosses the event horizon. This aspect, that the event horizon is merely a geometrical, rather than a physical, boundary, has observational consequences [2], but that is another story.

Despite the fact that gravitational collapse is a cataclysmic phenomenon wherein a multitude of physical processes (some understood, others yet to be discovered) are unleashed, the end-product – a black hole – is a pristine object. As Chandrasekhar says, “..... the only elements in the construction of black holes are our concepts of space and time. They are, thus, almost by definition, the most perfect macroscopic objects there are in the universe. And since the general theory of relativity provides a unique three-parameter family of solutions for their description, they are the simplest objects as well.” [3]

2. Black hole entropy

The simplicity and perfection of our conceptualization of black holes were dramatically challenged in the early seventies by Bekenstein [4] and Hawking [5], based on considerations

that germinate from the known quantum origin of all matter (and radiation). Starting with the simple observation that the area A_{hor} of the horizon of the simplest black hole – the Schwarzschild black hole – is a quadratic function of the mass M , Bekenstein [4] noted the incremental result (in units $G = c = 1$)

$$dM = \Theta dA_{hor}, \quad \Theta \equiv 1/4M. \quad (2)$$

The most general black hole in general relativity, viz., the Kerr-Newman solution, the ‘rationalized’ area of the event horizon is given by

$$A_{hor} = 4\pi(r_+^2 + L^2/M^2), \quad r_{\pm} \equiv M \pm \sqrt{M^2 - Q^2 - L^2/M^2}; \quad (3)$$

one then obtains similarly the incremental formula

$$dM \equiv \Theta dA_{hor} + \Phi dQ + \Omega \cdot dL, \quad (4)$$

where, $\Theta \equiv (r_+ - r_-)/4A_{hor}$, $\Phi \equiv 4\pi Qr_+/A_{hor}$, $\Omega \equiv 4\pi L/MA_{hor}$. Eqs. (2) and (4) formally resemble the First Law of thermodynamics

$$dU = TdS + PdV, \quad (5)$$

where the second term represents the work done on the system. If we attribute to the quantities Φ and Ω in (4) above the standard interpretation of electrostatic potential and angular velocity on the horizon, then the second and third terms in the r.h.s. of (4) also represent the work done on the black hole.

The analogy that seems to suggest itself is reinforced by Hawking’s theorem on black hole mechanics [6], that the horizon area of an isolated black hole never decreases in any transformation. *E.g.*, if two black holes of area A_1 and A_2 fuse together to form a black hole of area A_{1+2} , then the theorem asserts that $A_{1+2} > A_1 + A_2$. On the basis of these observations and results, Bekenstein made the bold proposal that a black hole *does* have an entropy S_{bh} proportional to the area of its horizon,

$$S_{bh} = \text{const.} \times A_{hor}. \quad (6)$$

This relationship, between a thermodynamic quantity and a geometric measure, is so striking that it warrants an interpretation. Drawing upon Shannon’s information theory [7] and Brillouin’s classic work relating it to thermodynamics [8], Bekenstein proposed an information theoretic connotation for S_{bh} .

Consider, for instance, the isothermal compression of an ideal gas. The thermal entropy of the gas certainly decreases due to the compression. However, one now has better information about the position of the molecules, as they are now more localized. In fact, according to Brillouin, the increase in information $\Delta I = -\Delta S$, where, ΔS is the decrease in entropy. It follows then that entropy measures lack of information about the internal configurations of the system. If p_n is the probability of occurrence of an internal configuration labelled by the positive integer n , the entropy is given by Boltzmann’s formula (with the Boltzmann constant $k_B = 1$)

$$S = - \sum p_n \ln p_n. \quad (7)$$

The smallest unit of information is the binary bit, with $n = 2$ and $P_n = \frac{1}{2}$; this corresponds to a maximum entropy of $\ln 2$, which might be taken to be a unit of entropy.

The black hole entropy S_{bh} arises from our lack of information about the nature of gravitational collapse. The post-collapse configuration is completely characterised by three parameters, viz., M, Q, L which encode in an unknown way the diverse set of events occurring during collapse, just as a thermodynamic system is characterised by a few quantities like pressure, volume, temperature etc. which encode the microstates of the system. Thus, the black hole entropy is not to be regarded as the thermal entropy inside the black hole horizon. As Bekenstein remarks, "In fact, our black hole entropy refers to the equivalence class of all black holes which have the same M, Q, L, \dots " [4]. In units where $G = c = 1$, the only way in which S_{bh} and A_{hor} can be proportional is if the constant of proportionality has the form η / \hbar , where η is a dimensional number of $O(1)$. The appearance of \hbar is "... a reflection of the fact that the entropy is ... a count of states of the system, and the underlying states are quantum in nature ... It would be somewhat pretentious to calculate the precise value of η without a full understanding of the quantum reality which underlies a 'classical' black hole" [4].

The 'quantum reality' Bekenstein refers to presumably subsumes quantum gravitational effects which inevitably occur in gravitational collapse. A complete quantum gravity theory which serves the purpose is still not available, although there are candidates with promise as we discuss in the sequel. These notwithstanding, a semiclassical estimate of η a la Bekenstein may be given [4]. Eq. (6) is first generalized to

$$S_{bh} = f(\alpha), \quad (8)$$

where, $\alpha \equiv A_{hor} / 4\pi$ and f is a monotonically increasing function. Using techniques of Christodolou, Bekenstein argues that the minimum increase in the area of a black hole due to an infalling particle of mass μ and size b is given by

$$(\Delta\alpha)_{min} = 2\mu b_{min}. \quad (9)$$

Now, $b_{min} = \lambda_c$, or R_{sch} , whichever is larger, where λ_c is the Compton wavelength of the particle and R_{sch} the Schwarzschild radius; for $\mu < (\hbar/2)^{1/2}$, $\lambda_c \geq R_{sch}$, and the other way round for $\mu > (\hbar/2)^{1/2}$. In the first case, $(\Delta\alpha)_{min} = 2\hbar$, and in the second case, $(\Delta\alpha)_{min} = 4\mu^2 > 2\hbar$. Thus, $(\Delta\alpha)_{min} = 2\hbar$, as is indeed the case for an 'elementary' particle. This then also quantifies the minimum loss of information due to the particle entering the black hole horizon. Recalling now that the minimum loss of information is a binary bit corresponding to an increase in entropy of $\ln 2$, one sets [4]

$$\frac{df}{d\alpha} (\Delta\alpha)_{min} = \ln 2, \quad (10)$$

obtaining

$$f(\alpha) = \frac{1}{2} \frac{\alpha}{\hbar} \ln 2 \quad \text{or} \quad \eta = \frac{1}{2} \ln 2. \quad (11)$$

Retrieving all factors of G, c and k_B , the Bekenstein formula for entropy, in conventional units, is

$$S_{bh} = \frac{1}{8\pi G\hbar} \ln 2 \, k_B c^3 A_{hor}. \quad (12)$$

This formula is identical (except for the factor of $\ln 2$ which one may think of as a choice of units of entropy) to the one proposed by Hawking [5] based on consistency with the rate of black hole radiation derived by him. We shall come to Hawking's work shortly ; prior to that two remarks are in order, following Bekenstein.

The first of these concerns the so-called black hole temperature, T_{bh} , defined in analogy with the temperature in thermodynamics : $T^{-1} = (\partial S / \partial U)_V$, here

$$T_{bh}^{-1} \equiv \left(\frac{\partial S_{bh}}{\partial M} \right)_{L,Q}, \quad (13)$$

whence

$$T_{bh} = \frac{2\hbar}{8\pi \ln 2} \Theta, \quad (14)$$

with $\Theta (= 1/4M$ for the Schwarzschild black hole) is the surface gravity of the black hole, a geometric quantity that remains constant over the event horizon. Once again, Bekenstein does not favour a *total* analogy with the standard notion of temperature in thermodynamics : "but we should emphasize that one should not regard T_{bh} as *the* temperature of the black hole ; such an identification can lead to paradoxes".

The second remark has to do with the Second Law of thermodynamics. When an object falls down a black hole, there is a decrease in the thermal entropy of the universe. But, as we have already seen, this is invariably accompanied by an increase in the area of the horizon, with a resultant increase in S_{bh} . This led Bekenstein [4] to propose a Generalized Second Law of thermodynamics for a universe with black holes : $S_{thermal} + S_{bh}$ *never decreases*.

As one of his motivations for his pioneering work on black hole radiation, Hawking considered the situation in which a black hole (with 'temperature' T_{bh}) is immersed in black body radiation with temperature $T < T_{bh}$. The question is, *does the black hole radiate* as it would if it were a thermal object at actual temperature T_{bh} ? Hawking's answer is of course an emphatic affirmative following his discovery that black holes do emit particles in a thermal spectrum at a temperature T_{bh} : ".... if one accepts that black holes do emit particles the identification of $T_{bh} = \hbar \Theta / 4\pi$ with the temperature of the black hole and $A_{hor} / 4\hbar$ with the entropy of the black hole is established, and a Generalized Second Law confirmed". It is clear that, notwithstanding Bekenstein's cautionary remarks, Hawking's view of black hole entropy (and temperature) is in fact very close to standard thermodynamic notions.

To summarize, the black hole entropy problem consists of identifying and counting the underlying quantum states in an attempt to verify if the 'quantum reality' actually bears out the semiclassical Bekenstein-Hawking formula, with the correct constant of proportionality, in the appropriate limit. Note that S_{bh} is in fact a bit different from the standard thermodynamic entropy which is usually a bulk quantity, expressed as a function of the volume of the system, rather the *area* of the boundary surface. Thus, it should suffice to focus on microstates associated with the event horizon (which, remember, is a boundary of spacetime as seen by an external observer).

We shall not survey various semiclassical approaches to this question, *i.e.*, approaches not relying in a quantum theory of gravity. These have been adequately reviewed in another

article in these proceedings [9]. In what follows, we consider two avenues of attack which are claimed to be theories of quantum gravity, namely, string theory and canonical quantum gravity. Although the string approach to the problem has also been nicely reviewed in another article in these proceedings [10], we shall make several brief remarks on that approach, some of a critical nature. The canonical quantum gravity approach will receive a more detailed discussion.

3. Quantum gravity

A theory of quantum gravity is supposed to describe nature at a length scale at which quantum mechanical effects and gravitational effects become of comparable strength. Thus, requiring that the Compton wavelength and the Schwarzschild radius of a particle of mass m to be of the same order, one deduces this length scale to be of order $(G\hbar/c^3)^{1/2}$ – a fundamental length scale first deduced by Planck. Numerically, this length is of the order 10^{-33} cm, corresponding to an energy of 10^{19} GeV. So far, there is no complete description of nature at such tiny lengths. Among the prospective candidates the most popular is (super) string theory [11].

A. String theory :

The basic postulate underlying this theory is that at a length scale $l_s \ll 10^{-15}$ cm, the universe is populated by massless relativistic strings propagating in a D dimensional flat Minkowski background spacetime. Quantum mechanical consistency of superstring dynamics requires that $D = 10$. All elementary particles of nature are essentially quantum excitations (modes) of the string with various masses (starting from zero) characterised by the string tension α'^{-1} where $\alpha' \sim l_s^2$. The massless spectrum of closed superstrings (loops) includes spin 2 gravitons, the quanta characterising small fluctuations of spacetime geometry around the classical Minkowski background. Graviton scattering amplitudes in superstring theory are believed to be finite to all orders in string perturbation theory [11], although a complete formal proof of this property is still not available.¹ A low energy effective local field theory exclusively for the gravitons can be obtained as a power series in derivatives (external momenta) from the four-graviton string amplitude restricted to small momenta. The first few terms in this series are identical, remarkably, to the first few terms in the expansion of the Einstein-Hilbert action around a flat Minkowski metric. However, (a) there is no evidence that the series derived from the string amplitude converges, and (b) even if it does, there is no guarantee that it will converge to the Einstein-Hilbert action. Notwithstanding these caveats, string theorists assume that general relativity in its entirety is derivable as a *perturbative expansion around a flat background*, and even ‘quantum’ corrections to it due to virtual effects of massive string modes can be computed. As a corollary, the Newton constant can be derived from string parameters : $G \sim \alpha' g_s^2$, where, g_s is the dimensionless coupling constant of superstring theory.

The hypothesis that ‘quantum gravity’ is obtained as a perturbative expansion around a (flat) classical background appears flawed on two counts. First of all, consider gravitational scattering of particles at squared centre-of-mass energy s and squared momentum transfer t ; two phenomenological dimensionless coupling parameters that appear in the amplitudes are Gs and Gt . The domain of quantum gravity – the Planckian regime – is characterised by both Gs and Gt being of order unity or larger. In other words, *quantum gravity is inherently non-perturbative*. At Planck scale, spacetime fluctuations are anything but small, and a perturbative

¹This finiteness of graviton amplitudes resolves the malaise that a *local field* theory of gravitons, without the massive modes associated with strings, invariably exhibits in the ultraviolet [12].

theory of small fluctuations can hardly suffice to describe them, with or without the paraphernalia of string theory. From a phenomenological standpoint, unlike quantum electrodynamics or chromodynamics, quantum 'gravitodynamics' does not seem to have a weak coupling domain. At sub-Planckian energies, Einstein's classical general relativity gives an excellent description of nature insofar as gravitation is concerned, thereby rendering a theory of gravitons, finite or not, physically irrelevant. Secondly, the key question of quantum gravitation is : what is the nature of spacetime geometry at the Planck scale ? Any theory formulated in terms of a non-dynamical classical background cannot possibly arrive at an answer, in as much as the quantum theory of the hydrogen atom cannot be formulated in terms of fluctuations around the Kepler problem for a Coulomb potential.

Let us, however, momentarily suspend these reservations, and follow the path of string theorists towards the issue of black hole entropy. In string theory, the assumption that string amplitudes yield Einstein's general relativity, coupled of course to other massless fields (dilaton, axion, Ramond-Ramond gauge fields) in the string spectrum, leads immediately to classical black hole solutions. These solutions are characterised by a larger parameter space than that pertaining to black holes of general relativity. In particular, non-rotating stringy black holes² carry a number of $U(1)$ (electric-like) charges, associated with solitonic string excitations (the so-called Ramond-Ramond states) of the $D = 10$ type II superstring theory (or $D = 11$ M-theory). Upon toroidal compactification of these theories down to five or four dimensions, these appear as the so-called D(-irichlet) branes, which carry R-R charges. Some non-rotating black holes of this theory are identified with certain (intersecting) D-brane states. The degeneracy of these particular D-brane states can be computed exactly in the weak coupling limit of the theory ($g_s \rightarrow 0$) as a function of the R-R charges. In the situation that these D-brane configurations saturate the Bogomol'nyi-Prasad-Sommerfield (BPS) bound $\sum_i Q_i = M_{bh}$, corresponding to *extremal* black holes, the entropy corresponding to this degeneracy *matches exactly* S_{bh} as calculated from the horizon area of the black holes using the Bekenstein-Hawking formula [14]. A clear understanding of the physics behind this dramatic correspondence is yet to emerge. Some plausibility arguments have nevertheless been advanced [15], [10].

These are based on the basic premise that the gravitational constant $G = \alpha' g_s^2$, so that, as g_s is fine tuned from weak to strong coupling, one passes from the string 'phase' to the black hole 'phase' for fixed α' . Alternatively, the ratio of the 'Schwarzschild radius' of the string to its length varies from small to $O(1)$ as g_s is tuned. The degeneracy formula for BPS states is argued to remain intact during the tuning of g_s , appealing to non-renormalization theorems of unbroken extended ($N > 1$) spacetime supersymmetry. Consequently, the entropy calculated from the degeneracy appears to agree with that of the black hole calculated from the semiclassical formula.³

A major shortcoming of the foregoing correspondence is its crucial dependence on unbroken supersymmetry, a property certainly untenable in the real world. What happens to

We confine our discussion here to stringy or M-theory black holes with a unique curvature singularity and regular horizons. Other stringy solutions with horizons that acquire curvature singularities in certain regions of parameter space [13] fall outside the scope of this review.

Ironically, the Bekenstein-Hawking formula is known to break down in semiclassical general relativity in the extremal situation in which the entropy of the black hole is argued to vanish [16], [17]. In this case, the extremal limit is the same as exact extremality. For stringy black holes exact extremality also yields a null entropy, following arguments of [16], while the extremal *limit* seems to agree with the Bekenstein-Hawking formula. Clearly this warrants a better understanding.

the degeneracy formula when supersymmetry is broken and BPS saturation is no longer valid as a property that survives quantum corrections, is an open question. Thus, it is a correspondence that works for a highly idealized situation, for a special kind of black holes, most likely very different from black holes seen in nature. The latter most likely would correspond to generic non-extremal general relativistic black holes characterised by a far smaller parameter space than that appearing in string theory. This is related to the necessary existence, in string theory, of spacetime dimensions beyond the four observed in the real world. A further technical problem is the computation of the degeneracy itself ; here the practice is to assume that the complicated intersecting D-brane configuration collapses to a 'long effective string' [18] as the entropically favoured configuration, and then use the machinery of the 2d conformal field theory corresponding to this string, with central charge $c = 6$. The trouble with this is that it is not always straightforward to derive this value of the central charge without extra ad hoc assumptions. This is the situation, *e.g.*, for four dimensional black holes obtained in type IIB superstring theory as intersecting 2-5-6 branes [15], where the 'long string' is non-trivial to identify.⁴ The area law would still emerge in this case, but only upto an overall constant.

B. Canonical quantum gravity :

This approach, also called Quantum General Relativity, envisages an exact solution to the problem of quantization of standard four dimensional general relativity, in contrast to the previous perturbative path around a flat classical background. In this respect, it is closer to the canonical quantum theory of the hydrogen atom, for instance.

The canonical treatment of classical general relativity, otherwise known as geometrodynamics [19], is traditionally formulated in terms of 3-metrics, *i.e.*, restrictions of the metric tensor to three dimensional spacelike hypersurfaces ('time slices'). Canonically conjugate variables to these are then constructed, Poisson brackets between them defined and the entire set of first class constraints derived. The problem with this is that the constraints remain quite intractable.

A significant departure from this approach is to formulate canonical general relativity as a theory of 'gauge' connections, rather than 3-metrics [20]. Some of the constraints simplify markedly as a consequence, allowing exact treatment, although this is not true for all the constraints (*e.g.*, the Hamiltonian constraint still remains unsolvable). The method has also undergone substantial evolution since its inception ; a complex one-parameter family of connection variables is available as one's choice of the basic 'coordinate' degrees of freedom. The original Ashtekar choice [20], *viz.*, the self-dual $SL(2, C)$ connection (inspired by work of Sen [21]), corresponding to one member of this family, is 'geometrically and physically well-motivated' because the full tangent space group then becomes the gauge group of the canonical theory [22]. However, quantizing a theory with complex configuration degrees of freedom necessitates the imposition of subsidiary 'reality' conditions on the Hilbert space, rendering the formulation unwieldy.

A better alternative, related to the former by canonical transformations, is to deal with the Immirzi family of *real* $SU(2)$ connections confined to the time-slice M

$$A_i^{(\beta)a} \equiv \epsilon^{abc} \Omega_{i bc} + \beta g_{ij} \Omega^j_{0a} , \quad (15)$$

labelled by a positive real number β ($\sim O(1)$) known as the Immirzi parameter [22].⁵ This yields the Immirzi family of curvatures (restricted to M),

$$F_{ij}^{(\beta)a} \equiv \partial_{[i} A_{j]}^{(\beta)a} + \epsilon^{abc} A_i^{(\beta)b} A_j^{(\beta)c}. \quad (16)$$

The variables canonically conjugate to these are given by the so-called solder form

$$E_i^{(\beta)a} = \frac{1}{\beta} \epsilon^{abc} \epsilon_{ijk} e_j^b e_k^c. \quad (17)$$

The canonical Poisson bracket is then given by

$$\{A_i^{(\beta)a}(x), E_j^{(\beta)b}(y)\} = \beta \delta_{ij} \delta^{ab} \delta(x, y). \quad (18)$$

Canonical quantization in the connection representation implies that physical states are gauge invariant functionals of $A_i^{(\beta)a}(x)$ and

$$E_i^{(\beta)a} \rightarrow \hat{E}_i^{(\beta)a} \equiv \frac{\beta \hbar}{i} \frac{\delta}{\delta A_i^{(\beta)a}}. \quad (19)$$

A useful basis of states for solution of the quantum constraints are the ‘spin network’ states which generalize the loop space states used earlier [23]. A spin network consists of a collection *edges* and *vertices*, such that, if two distinct edges meet, they do so in a vertex. It is a lattice whose edges need not be rectangular, and indeed may be non-trivially knotted. *E.g.*, the graph shown in Figure 2 has 9 edges and 6 vertices.

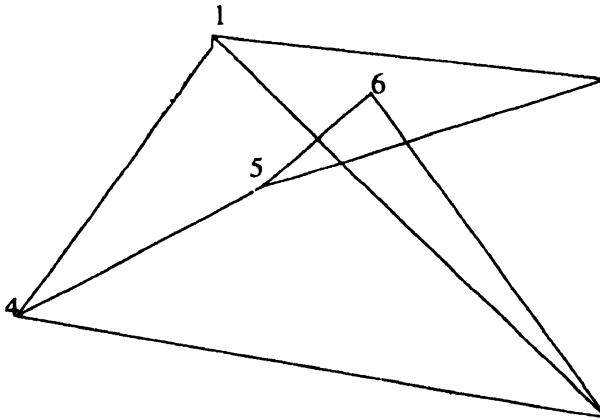


Figure 2. A spin network with 9 edges and 6 vertices

To every edge g_i ($i = 1, 2, \dots, 9$) we assign a spin J_i which takes all half-integral values including zero. Thus, each edge transforms as a finite dimensional irreducible representation of $SU(2)$. In addition, one assigns to each edge a Wilson line functional of the gauge connection $h_i(A) = P \exp \int_i d\gamma_i^i (A \cdot \tau)$, where τ^a are $SU(2)$ generators in the adjoint representation. To every vertex is assigned as $SU(2)$ invariant tensor C^v . These assignments completely define

⁵ Here Ω_{μ}^{bc} is the standard Levi-Civita connection, $i, j = 1, 2, 3$ are spatial world indices, and $a, b, c = 1, 2, 3$ are spatial tangent space indices, and g_{ij} is the 3-metric.

the basis states, which form a dense set in the Hilbert space of gauge invariant functionals of $\mathcal{B}A$. The inner product of these states then induces a measure on the space of connections which can be used to define a 'loop transform' [22] of physical states, representing the same state, by diffeomorphism invariance. 'Weave' states, supported on complicated and fine meshed nets (with meshes of Planck scale size) are supposedly typical physical states. Thus, the classical spacetime continuum metamorphoses in the quantum domain into a space of 'weaves' with meshes of Planck scale size on which all curvature (and indeed all dynamics) is concentrated. The Einsteinian continuum emerges when we view the weaves from afar, and are no longer able to see the meshes.

Observables on the space of physical states (like the weaves) include geometrical operators like the area and volume operators, which typically are functionals of the canonical variables. To calculate the spectrum of these operators in the connection representation requires a technique of 'regularization' since the classical definition of these quantities translates into singular objects upon naive quantization. *E.G.*, the area operator $\hat{A}(S)$ corresponding to a two dimensional surface S intersecting a subset \mathcal{L} of edges of a net, not touching any of its vertices and having no edge lying on S is formally defined as

$$\hat{A}(S)\psi_n \equiv \left(\int d^2\sigma \sqrt{n_i n_j \hat{E}^{ia} \hat{E}^{jb}} \right)_{\text{reg}} \psi_n \quad (20)$$

For large areas compared to l_{Planck}^2 , this reduces to [24], [25]

$$\hat{A}(S)\psi_n = \beta \hbar l_{\text{Planck}}^2 \sum_{l \in \mathcal{L}} \sqrt{J_l(J_l + 1)} \psi_n. \quad (21)$$

The discreteness in the eigen-spectrum of the area operator is of course reminiscent of discrete energy spectra associated with stationary states of familiar quantum systems. Each element of the discrete set in (21) corresponds to a particular *number* of intersections ('punctures') of the spin net with the boundary surface S . Diffeomorphism invariance ensures the irrelevance of the locations of punctures. This will have important ramifications later.

We now consider the application of the foregoing formalism to the calculation of entropy of the four dimensional Schwarzschild black hole, following [24-28]. The basic idea is to concentrate on the horizon as a boundary surface of spacetime (the rest of the boundary being described by the asymptotic null infinities \mathcal{I}^\pm), on which are to be imposed boundary conditions specific to the horizon geometry of the Schwarzschild black hole vis-a-vis its symmetries *etc.* These boundary conditions then imply a certain description for the quantum degrees of freedom on the boundary. The entropy is calculated by counting the 'number' of boundary degrees of freedom. The region of spacetime useful for our purpose is depicted in the Penrose and Finkelstein diagrams in Figure 3. The four-fold M has as a boundary the event horizon H in addition to \mathcal{I}^\pm . Δ is a 'finite patch' on H of constant cross-sectional area A_S . M is a particular time-slice which intersects H (in particular Δ) in the 2-sphere S .

Standard asymptotically flat boundary conditions are imposed on \mathcal{I}^\pm ; those on the event horizon essentially subsume the following: first of all, the horizon is a null surface with respect to the Schwarzschild metric; second, the black hole is an isolated one with no gravitational radiation on the horizon; thirdly, the patch Δ has two flat (angular) coordinates spanning a special 2-sphere which *coincides* with S , the intersection of the time-slice M with Δ .

The last requirement follows from the spherical symmetry of the Schwarzschild geometry. These boundary conditions have a crucial effect on the classical Hamiltonian structure of the theory, in that, in addition to the bulk contribution to the area tensor of phase space (the

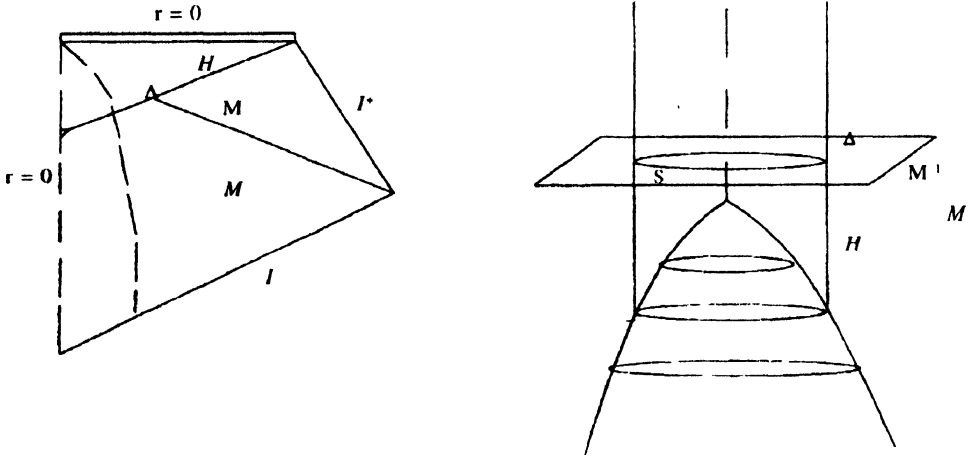


Figure 3. Penrose and Finkelstein diagrams showing the Schwarzschild black hole in the canonical framework

symplectic structure) arising in canonical general relativity, there is a *boundary* contribution. Notice that the boundary of the spacelike hypersurface M intersecting the black hole horizon is the 2-sphere S . Thus, the symplectic structure is given by

$$\begin{aligned} \Omega|_{A^{(\beta)}, I^{(\beta)}} & \left(\delta E^{(\beta)}, \delta A^{(\beta)}; \delta E^{(\beta)'}, \delta A^{(\beta)'} \right) \\ &= \frac{1}{8\pi G} \int_M \text{Tr} \left(\delta E^{(\beta)} \wedge \delta A^{(\beta)'} - \delta E^{(\beta)'} \wedge \delta A^{(\beta)} \right) \\ & \quad - \frac{k}{2\pi} \int_{S=\partial M} \text{Tr} \left(\delta A^{(\beta)} \wedge \delta A^{(\beta)'} \right), \end{aligned} \quad (22)$$

where, $k \equiv \frac{\gamma_\Lambda}{2\pi\beta G}$. The second term in (22) corresponds to the boundary contribution to the symplectic structure; it is nothing but the symplectic structure of an $SU(2)$ level k Chern Simons theory living on M . This is consistent with an extra term that arises due to the boundary conditions in the action, that is exactly an $SU(2)$ level k Chern Simons action on the three dimensional piece Δ of the event horizon [28]. As a consequence of the boundary Chern Simons term, the curvature pulled back to S is proportional to the pullback (to S) of the solder form

$$F^{(\beta)} + \frac{2\pi\beta}{A_S} E^{(\beta)} = 0. \quad (23)$$

This is a key relation for the entropy computation as we now proceed to demonstrate.

In the quantum theory, we have already seen that spacetime ‘in the bulk’ is described by spin nets ($\{\psi_v\}$ say) at fixed time-slices. It has been shown [25] that spin network states

constitute an eigen-basis for the solder form with a discrete spectrum. Now, in our case, because of the existence of the event horizon which forms a boundary of spacetime, there are additional surface states $\{\psi_S\}$ associated with Chern Simons theory. In the canonical framework, the surface of interest is the 2-sphere S which forms the boundary of M . Thus, typically a state vector in the Hilbert space H would consist of tensor product states $\psi_v \otimes \psi_S$. Eq. (23) would now act on such states as an operator equation. It follows that the surface states $\{\psi_S\}$ would constitute an eigenbasis for $F^{(\beta)}$ restricted to S , with a discrete spectrum. In other words, *the curvature has a support on S only at a discrete set of points – punctures*. These punctures are exactly the points on S which are intersected by edges of spin network ‘bulk’ states, in the manner discussed earlier for the definition of the area operator. At each puncture p therefore one has a specific spin J_p corresponding to the edge which pierces S at p . The black hole can then be depicted (in an approximate sense) as shown in Figure 4.

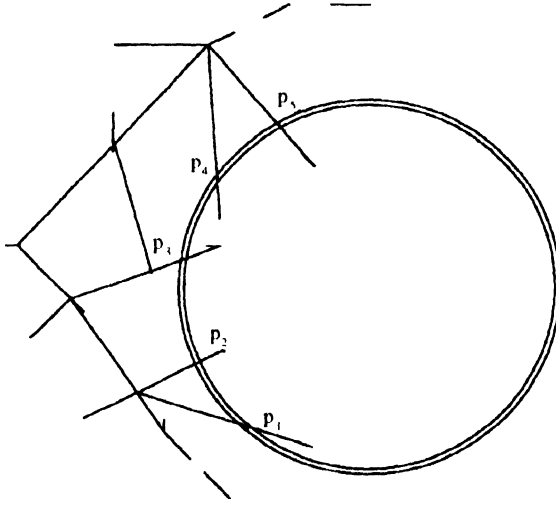


Figure 4. The Schwarzschild black hole in the spin network picture, showing five of the punctures $p_1 \dots p_5$

Consider now a set of punctures $P_{(n)} = \{p_1, J_{p_1}; p_2, J_{p_2}; \dots; p_n, J_{p_n}\}$. For every such set, there is a subspace H_V^P of H_V which describes the space of spin net states corresponding to the punctures. Similarly, there is a subspace H_S^P of H_S describing the boundary Chern Simons states corresponding to the punctures in P . The full Hilbert space is given by the direct sum, over all possible possible sets of punctures, of the direct product of these two Hilbert (sub) spaces, *modulo* internal gauge transformations and diffeomorphisms.⁶ Now, given that the Hamiltonian constraint cannot be solved exactly, one assumes that there is at least one solution of the operator equation acting on the full Hilbert space, for a given set of punctures P .

One now assumes that it is only the surface states ψ_v that constitute the microstates contributing to the entropy of the black hole S_{bh} , so that the volume states ψ_v are traced over, to yield the black hole entropy as

$$S_{bh} = \ln \sum \dim H_S^P. \quad (24)$$

The latter symmetry, in particular, as already mentioned, implies that the location of punctures on S cannot have any physical significance.

The task has thus been reduced to computing the number of $SU(2)_k$ Chern Simons boundary states for a surface with an area that as A_S to within $O(l_{Plank}^2)$. One now recalls a well-known correspondence between the dimensionality of the Hilbert space of the Chern Simons theory and the number of conformal blocks of the two dimensional conformal field theory (in this case $SU(2)_k$ Wess-Zumino-Witten model) 'living' on the boundary [30]. This correspondence now simplifies the problem further to the computation of the number of conformal blocks of the WZW model. Thus, *the problem of counting the microstates contributing to the entropy of a 4d Schwarzschild black hole has metamorphosed into counting the number of conformal blocks for a particular 2d conformal field theory.*

This number can be computed in terms of the so-called fusion matrices N_{ij}^r [31]

$$N^P = \sum_{\{r_i\}} N_{j_1 j_2}^{r_1} N_{r_1 j_3}^{r_2} N_{r_2 j_4}^{r_3} \dots N_{r_{p-2} j_{p-1}}^{r_{p-1}} \quad (25)$$

This is very similar to the composition of angular momentum in ordinary quantum mechanics ; it has been extended here to the infinite dimensional affine Lie algebra $SU(2)_k$. Diagrammatically, this can be represented as shown in Figure 5 below.

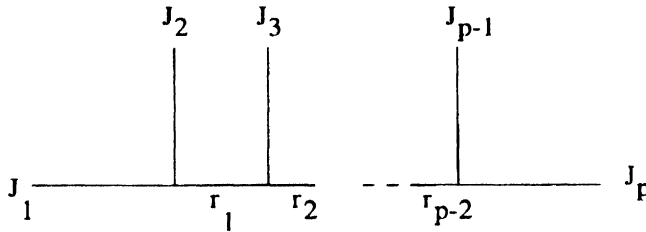


Figure 5. Diagrammatic representation of composition of spins J_i for $SU(2)_k$

Here, each matrix element N_{ij}^r is 1 or 0, depending on whether the primary field $[\phi_r]$ is allowed or not in the conformal field theory fusion algebra for the primary fields $[\phi_i]$ and $[\phi_j]$ ($i, j, r = 0, 1/2, 1, \dots, k/2$):

$$[\phi_i] \otimes [\phi_j] = \sum_r N_{ij}^r [\phi_r] \quad (26)$$

Eq. (25) gives the number of conformal blocks with spins j_1, j_2, \dots, j_p on p external lines and spins r_1, r_2, \dots, r_{p-2} on the internal lines.

We next take recourse to the Verlinde formula [31]

$$N_{ij}^r = \sum_s \frac{S_{is} S_{js} S_s^{tr}}{S_{0s}}, \quad (27)$$

where, the unitary matrix S_{ij} diagonalizes the fusion matrix. Upon using the unitarity of the S -matrix, the algebra (25) reduces to

$$N^P = \sum_{r=0}^{k/2} \frac{S_{j_1 r} S_{j_2 r} \dots S_{j_p r}}{(S_{0r})^{p-2}} \quad (28)$$

Now, the matrix elements of S_{ij} are known for the case under consideration ($SU(2)_k$ Wess-Zumino model) ; they are given by

$$S_{ij} = \sqrt{\frac{2}{k+2}} \sin \left(\frac{(2i+1)(2j+1)\pi}{k+2} \right) \quad (29)$$

where, i, j are the spin labels, $i, j = 0, 1/2, 1, \dots, k/2$. Using this S -matrix, the number of conformal blocks for the set of punctures P is given by

$$N^P = \frac{2}{k+2} \sum_{r=0}^{k/2} \prod_{l=1}^p \frac{\sin \left(\frac{(2jl+1)(2r+1)\pi}{k+2} \right)}{\sin \frac{(2r+1)\pi}{k+2}} \quad (30)$$

In the notation of [28], eq. (30) gives the dimensionality, $\dim H_S^P$, for arbitrary area of the horizon k and arbitrary number of punctures. The dimensionality of the space of states H_S of CS theory on three-manifold with S^2 boundary is then given by summing N^P over all sets of punctures P , $N_{bh} = \sum_P N^P$. Then, the entropy of the black hole is given by $S_{bh} = \log N_{bh}$.

Observe now that eq. (30) can be rewritten, with appropriate redefinition of dummy variables and recognizing that the product can be written as a multiple sum,

$$N^P = \left(\frac{2}{k+2} \right) \sum_{l=1}^{k+1} \sin^2 \theta_l \sum_{m_1=-l_1}^{l_1} \dots \sum_{m_p=-l_p}^{l_p} \exp \left\{ 2i \left(\sum_{n=1}^p m_n \right) \theta_l \right\}, \quad (31)$$

where, $\theta_l \equiv \pi l / (k+2)$. Expanding the $\sin^2 \theta_l$ and interchanging the order of the summations, a few manipulations then yield

$$N^P = \sum_{m_1=-l_1}^{l_1} \dots \sum_{m_p=-l_p}^{l_p} \delta_{(\sum_{n=1}^p m_n)_0} - \frac{i}{2} \delta_{(\sum_{n=1}^p m_n)_1} - \frac{i}{2} \delta_{(\sum_{n=1}^p m_n)_{-1}} \quad (32)$$

where we have used the standard resolution of the periodic Kronecker deltas in terms of exponentials with period $k+2$,

$$\delta_{(\sum_{n=1}^p m_n)_m} = \left(\frac{1}{k+2} \right) \sum_{l=0}^{k+1} \exp \left\{ 2i \left[\left(\sum_{n=1}^p m_n \right) - m \right] \theta_l \right\} \quad (33)$$

Notice that the explicit dependence on $k+2$ is no longer present in the exact formula (32).⁷

The foregoing calculation does not assume any restrictions on k or number of punctures p . Eq. (32) is thus an exact formula for the quantum entropy of a Schwarzschild black hole of any size, not necessarily large compared to the Planck size. However, there is no corresponding calculation of the spectrum of the area operator for arbitrary-sized surfaces ; the formula (21) is valid only for $A_S \gg l_{\text{Planck}}^2$. This is what should be delineated as the semiclassical limit of the

⁷ A similar method has been advocated in ref [32] for self-dual boundary conditions

theory, a domain in which true quantum gravitational effects do not yet make their appearance. In this restricted regime, our result (30) reduces to

$$N^P \sim \prod_{l=1}^p (2_{jl} + 1) \quad (34)$$

in agreement with the result of ref. [28]. It is not difficult to show that, upon performing the sum over all possible punctures, for large number of punctures and large areas ($k \gg 1$), $S_{bh} = A_S / 4G\hbar$ for a specific value of the Immirzi parameter [28]. To see if the B-H formula relating entropy with area is valid even when this restriction of large horizon area is lifted, as it would be in the full quantum theory, one needs to obtain the eigenvalues of the area operator without any assumptions about their size. This might entail a modified regularized area operator which measures horizon area in the quantum theory and is, in general, a constant of motion, *i.e.*, commutes with the Hamiltonian constraint. The completion of this part of the task should reveal quantum corrections to the semi-classical B-H formula.

The one ambiguity that has remained throughout the calculation of the entropy of the Schwarzschild black hole is the Immirzi parameter. This was 'fixed' by the requirement that the microstate counting actually gives the area law with the Hawking value of the proportionality constant. It has been claimed that this choice is universal for non-rotating black holes [28]. Recall that the existence of this parameter is independent of whether the spacetime one is quantizing is a black hole or not ; it is an inherent aspect of canonical quantum gravity. Therefore, it is not unlikely that it will actually be determined (may be through an eigenvalue equation) from the Hamiltonian constraint, when one gets a better understanding of that constraint [22].

It appears that methods of two dimensional conformal field theory effectively describe *quantitative* quantum physics of the black holes in four spacetime dimensions. In this respect, the similarity with the computational method adopted in the string theoretic approach, is quite remarkable, although the two conformal field theories used in the two paths remain quite different.⁸ Black hole entropy appears to be a global property that requires a counting of microstates of the horizon ; in the canonical quantum gravity case, this number is most likely a topological quantity, as it counts the boundary states of a three dimensional topological field theory. The calculation of the degeneracy of the 'long string' states in the D-brane approach might also share this property. In this respect, mention should be made perhaps of the ideas of 't Hooft and Susskind, on attempts to model the horizon as a 'hologram' [35], [36]. These ideas were very likely forerunners of the more recent realization that 'horizon properties' of realistic four dimensional black holes can be computed on the basis of two dimensional (conformal) field theories. The commonality that one observes between the two approaches with premises that have almost nothing to do with each other is indeed striking.

There is however one major flaw in both approaches ; the black hole curvature singularity seems to play no role at all. Thus, the two methods of computing number of microstates go through for any non-trivial boundary of spacetime (like a horizon), even though there may not be any spacetime singularity beyond that boundary.⁹ By concentrating only on possible properties of the horizon, an essential aspect of black holes is probably being ignored. In other

⁸ Although this seemed to be the case for the three dimensional BTZ black hole as well [33], it has now been established [34] that they are indeed the same if looked at more carefully.

⁹ See, *e.g.*, ref. [37] on this issue

words, none of these two approaches seems to illumine, in any manner whatsoever, the quantum nature of gravitational collapse. It stands to reason that any fundamental theory of quantum gravity will have to deal directly with spacetime singularities, just as quantum electrodynamics resolves the singularity problems of the Maxwell theory.

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